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Convergence of Discrete Laplace-Beltrami Operators Over Surfaces

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Abstract—The convergence property of the discrete Laplace-Beltrami operator is the foundation of convergence analysis of the numerical simulation process of some geometric partial differential equations which involve the operator. The aim of this paper is to review several already-used discrete Laplace-Beltrami operators over triangulated surface and study numerically, as well as theoretically, their convergent behavior. We show that none of them is convergent in general, but two of them, proposed by Desbrun *et al.* and Meyer *et al.*, are convergent in a special case. We point out that this special case is very important in the numerical simulation of geometric partial differential equations.

Keywords—Laplace-Beltrami operator, Surface triangulation, Convergence.

1. INTRODUCTION

Laplace-Beltrami operator, abbreviated as LB operator in this paper, is a generalization of the Laplacian from flat spaces to manifolds. LB operator plays a central role in many areas, such as image processing (see [1–4]), signal processing (see [5,6]), surface processing (see [7–12]), and the study of geometric partial differential equations (PDE) (see [1,3,13,14]). For instance, the mathematical formulation of the mean curvature flow, surface diffusion flow (see [14]) and Willmore flow (see [15]), etc., involves the first- and second-order LB operators. In solving numerically PDE, which involves the Laplacian, a standard technique is to approximate the operator by a finite divided difference operator. Likewise, the LB operator needs to be discretized in solving the geometric PDEs numerically. However, due to the complexity and the diversity of the discretized surfaces, the discretization of LB operator is not as simple as the Laplacian over the flat surface. In the literature, several discretizations of LB operator over surfaces have been proposed and used. However, to the best of author's knowledge, none of these discretizations has been proved to be convergent as the size of surface mesh goes to zero.

The convergence of the discrete LB operators is the foundation for the convergence of some numerical simulation process of PDE which involves the LB operator. The aim of this paper is to

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review several already-used discrete LB operators over triangulated surface and study numerically, as well as theoretically, their convergent behavior. We focus our attention on a family of discrete LB operators over triangulated surfaces, including Taubin's discretization (see [5,6]), Fujiwara's discretization (see [16]), Desbrun's *et al.* discretization (see [9]), Mayer's discretization (see [14]) and Meyer's *et al.* discretization (see [10]). All these discretizations are in the same form

$$\Delta_M f(p_i) = \sum_{j\in N(i)} w_{ij}\left(f(p_j) - f(p_i)\right), \tag{1.1}$$

where p_i and p_j are the vertices of the surface triangulation M, $N(i)$ is the index set of one-ring neighbors of vertex p_i, and w_{ij} are some positive constants.

It is known that LB operator relates closely to the mean curvature normal (see (2.2)). Hence, an approximation of mean curvature normal may lead to a discretization of the LB operator. On the approximation of curvatures, there exist also many approaches, such as the ones proposed by Chen, Hamann and Taubin to name a few [17–19]. However, these approaches do not yield the form (1.1), which is discussed in this paper.

The rest of the paper is organized as follows. In Section 2, we describe in more detail the discretizations of LB operator mentioned above, and then, in Section 3, we show numerically the convergence/unconvergence property of these discrete LB operators. In Section 4, we give several theoretical results of the convergence for those discrete LB operators which converge in the numerical experiment. The proofs of these results are given in Section 5. Section 6 concludes the paper.

2. LB OPERATOR AND ITS DISCRETIZATION

Let $\mathcal{M} \subset \mathbb{R}^3$ be a two-dimensional manifold, and $\{U_\alpha, x_\alpha\}$ be the differentiable structure. The mapping x_α with $x \in x_\alpha(U_\alpha)$ is called a parameterization of $\mathcal{M}$ at x. Denoting the coordinate U_α as (ξ_1, ξ_2), then, for $f \in C^2(\mathcal{M})$, the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$ applying to f is given by (see [20, page 43])

$$\begin{aligned}\Delta_{\mathcal{M}} f &= \frac{1}{\sqrt{\det(G)}} \sum_{ij} \frac{\partial}{\partial \xi_i}\left(g^{ij}\sqrt{\det(G)}\frac{\partial f}{\partial \xi_j}\right)\\ &= \frac{1}{\sqrt{\det(G)}}\left[\frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}\right]\sqrt{\det(G)}G^{-1}\left[\frac{\partial f}{\partial\xi_1}, \frac{\partial f}{\partial\xi_2}\right]^{\mathrm{T}},\end{aligned} \tag{2.1}$$

where g^{ij} is defined by $(g^{ij})_{ij} := G^{-1}$,

$$G^{-1} = \frac{1}{\det(G)}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{bmatrix}, \quad G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{bmatrix}, \qquad g_{ij} = \langle t_i, t_j\rangle,$$

and $t_i = \frac{\partial x}{\partial \xi_i}$ are the tangent vectors. From (2.1), we can see that

$$\Delta_{\mathcal{M}}(\alpha f) = \alpha\Delta_{\mathcal{M}} f, \qquad \Delta_{\mathcal{M}}(f+g) = \Delta_{\mathcal{M}} f + \Delta_{\mathcal{M}} g.$$

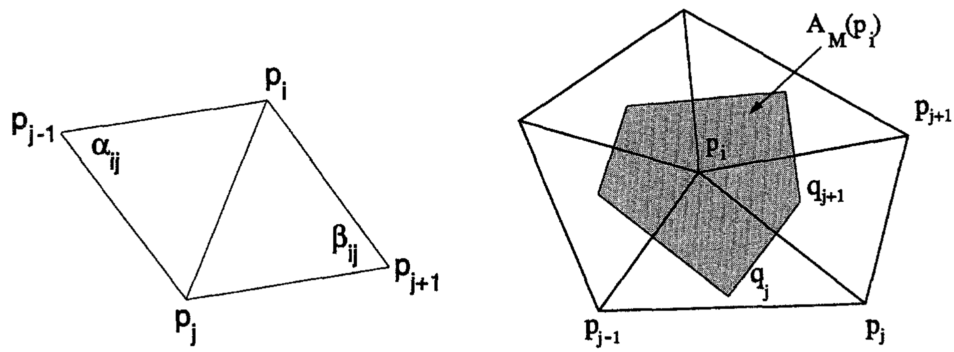
Let p be a surface point of $\mathcal{M}$. Then, it is known that (see [21, page 151])

$$\Delta_{\mathcal{M}} p = 2H(p) \in \mathbb{R}^3, \tag{2.}$$

where $H(p)$ is the mean curvature normal at p, i.e., $\|H(p)\|$ is the mean curvature, $H(p)/\|H(p)\|$ is the unit surface normal. Now, we consider the discretization of $\Delta_{\mathcal{M}} p$.

Let M be a triangulation of surface $\mathcal{M}$. Let $\{p_i\}_{i=1}^N$ be the vertex set of M. The discretized $\Delta_{\mathcal{M}} p$ considered in this paper is in the following form

$$\Delta_M p_i = \sum_{j\in N(i)} w_{ij}\left(p_j - p_i\right), \tag{2.3}$$



(a) The definition of the angles α_{ij} and β_{ij}. (b) The definition of the area $A(p_i)$.

Figure 1.

where $N(i)$ is the index set of one-ring neighbors of vertex p_i, w_{ij} are some positive constants. For a function f on surface $\mathcal{M}$, the discretization of $\Delta_{\mathcal{M}} f$ is (1.1), correspondingly.

1. Taubin's *et al.* Discretization (see [5,6,9,22])

This is a class of discretizations in the following form

$$\Delta_M^{(1)} f(p_i) = \sum_{j \in N(i)} w_{ij} \left(f(p_j) - f(p_i)\right) \tag{2.4}$$

where the weights w_{ij} are positive numbers and $\sum_{j\in N(i)} w_{ij} = 1$. There are several ways to determine the weights. A simple way is to take $w_{ij} = 1/|N(i)|$, where $|\cdot|$ denotes the cardinality of a set. A more general way is to define them by a positive function ϕ

$$w_{ij} = \frac{\phi(p_i, p_j)}{\sum_{k\in N(i)} \phi(p_i, p_k)},$$

and function $\phi(p_i, p_j)$ can be the surface area of the two faces that share the edge $[p_i, p_j]$, or some power of the length of the edge: $\phi(p_i, p_j) = \|p_i - p_j\|^\alpha$. Fujiwara take $\alpha = -1$ (see [16]). Desbrun's *et al.* (see [9]) define w_{ij} as

$$w_{ij} = \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{\sum_{k\in N(i)} \cot\alpha_{ik} + \cot\beta_{ik}},$$

where α_{ij} and β_{ij} are the triangle angles as shown in Figure 1a. Polthier's discretization (see [22]) is similar to the one given by Desbrun *et al.* (see [9]). He takes

$$w_{ij} = \frac{1}{2}\left(\cot\alpha_{ij} + \cot\beta_{ij}\right),$$

without imposing the normalization condition $\sum w_{ij} = 1$.

It is easy to see that discretization (2.4) could not be an approximation of $\Delta_{\mathcal{M}}$, since $\Delta_M p_i \to 0$ as the size of the surface mesh goes to zero. Hence, we shall not consider the convergence property of these discretization in the following.

2. Mayer's Discretization (see [14]).

Let $D_\epsilon(z)$ be a small disk at a point z on the surface $\mathcal{M}$. Then, for a sufficiently smooth function f defined on the surface, we have, by Green's formula

$$\int_{D_\epsilon(z)} \Delta_{\mathcal{M}} f(x)\, dx = \int_{\partial D_\epsilon(z)} \partial_\nu f(s)\, ds, \tag{2.5}$$

where ν is the intrinsic outer normal of the boundary of the disk, it is tangential to the surface. Discretizing (2.5) at p_i over the triangular surface mesh M, Mayer got the following approximation.

$$\Delta_M^{(2)} f(p_i) = \frac{1}{A(p_i)} \sum_{j \in N(i)} \frac{\|p_{j'} - p_j\| + \|p_{j''} - p_j\|}{2} \frac{f(p_j) - f(p_i)}{\|p_j - p_i\|}, \tag{2.6}$$

where $A(p_i)$ is the sum of areas of triangles around p_i, $j', j'' \in N(i) \cap N(j)$. We can see that (2.6) is derived from (2.5) by approximating $\int_{D_\epsilon(z)} \Delta_{\mathcal{M}} f(x)\, dx$, $\partial_\nu f(s)$ and ds with $\Delta_M^{(2)} f(p_i) A(p_i)$, $\frac{f(p_j)-f(p_i)}{\|p_j-p_i\|}$ and $\frac{\|p_{j'}-p_j\|+\|p_{j''}-p_j\|}{2}$, respectively. Hence, $\Delta_M^{(2)}$ is an approximation of $\Delta_{\mathcal{M}}$.

3. Desbrun's *et al.* Discretization (see [9,10])

From a differential geometry definition of mean curvature normal, one has

$$\lim_{diam(\mathcal{A}) \to 0} \frac{3\nabla \mathcal{A}}{2\mathcal{A}} = -H(p),$$

where $\mathcal{A}$ is the area of a small region around point p, where the curvature is needed, and ∇ is the gradient with respect to the (x, y, z) coordinates of p. From (2.9), Desbrun *et al.* get the following discretization,[1]

$$\Delta_M^{(3)} f(p_i) = \frac{3}{A(p_i)} \sum_{j \in N(i)} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} \left[f(p_j) - f(p_i) \right],$$

where $N(i)$ is the index set of 1-ring neighbor vertices of vertex p_i, α_{ij} and β_{ij} are the triangle angles as shown in Figure 1a, $A(p_i)$ is the sum of areas of triangles surrounding vertex p_i. (2.8) could be derived easily from (2.7), by writing $A(p_i)$ in the following form.

$$A(p_i) = \sum_{j \in N(i)} \frac{1}{2} \sqrt{\|p_j - p_i\|^2 \|p_{j+1} - p_i\|^2 - (p_j - p_i, p_{j+1} - p_i)^2},$$

and then, taking partials of $A(p_i)$ with respect to the coordinates of p_i.

4. Meyer's *et al.* discretization (see [10])

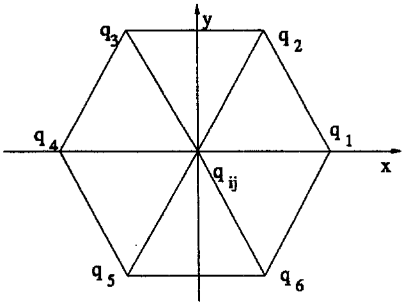
$$\Delta_M^{(4)} f(p_i) = \frac{1}{A_M(p_i)} \sum_{j \in N(i)} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} \left[f(p_j) - f(p_i) \right], \tag{2.9}$$

where α_{ij} and β_{ij} are defined as before, $A_M(p_i)$ is an area for vertex p_i as shown in Figure 1b, where q_j is the circumcenter point for the triangle $[p_{j-1}, p_j, p_i]$, if the triangle is nonobtuse. If the triangle is obtuse, q_j is chosen to be the midpoint of the edge opposite to the obtuse angle.

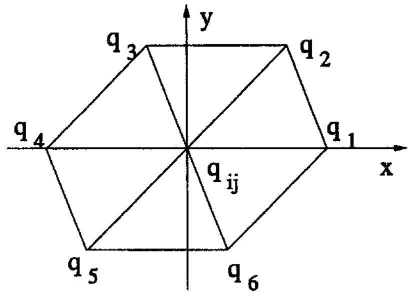
3. NUMERICAL EXPERIMENTS

The aim of this section is to exhibit the numerical behaviors of the discrete LB operators defined by (2.6)–(2.9), and determine which of them is numerically convergent to the exact value. Let $\{M_i\}$ be a sequence of triangulation of a surface $\mathcal{M}$. Let P_i be the vertex set of M_i. If $P_i \subset P_{i+1}$, then, we say the triangulation $\{M_i\}$ is hierarchical. The maximal edge length h_{M_i} of M_i is called the mesh size of M_i.

[1](2.7) and (2.8) differ from Desbrun *et al.*'s by a factor -3.



(a)



(b)

Figure 2. The triangulations of the domain.

DEFINITION 3.1. *Let $\{M_i\}$ be a hierarchical triangulation sequence of a smooth surface $\mathcal{M}$. Let Δ_{M_i} be a discrete LB operator defined on M_i, and the mesh size h_{M_i} go to zero as $i \to \infty$. Then, we say the discrete LB operator Δ_{M_i} is convergent if*

$$\lim_{i \geq j,\, i \to \infty} \Delta_{M_i} p = \Delta_{\mathcal{M}} p, \qquad \forall p \in P_j.$$

To show the numerical convergence of the discrete LB operators, we take several two-variable functions over xy-plane as surfaces in $\mathbb{R}^3$, so that the exact mean curvatures can be computed. Both the exact and approximated mean curvatures are computed at some selected domain points $q_{ij} = (x_i, y_j)$. These points are chosen as $(x_i, y_j) = (i/20, j/20)$ for $i = 1, \ldots, 19$, $j = 1, \ldots, 19$. The surfaces are triangulated around q_{ij} by triangulating the domain first, and then, mapping the planner triangulation onto the surfaces by the selected bivariate functions. As a first and simple case, the domain around q_{ij} is triangulated locally as shown in Figure 2a, where

$$q_k = q_{ij} + r\left(\cos\theta_k, \sin\theta_k\right), \qquad \theta_k = (k-1)\,\pi/3, \qquad k = 1, \ldots, 6, \tag{3.1}$$

and $r = 1/n$. The convergence property and the convergence rate are checked by taking $n = 8, 16, 32, \ldots$. The functions we use are the following.

$$\begin{aligned}
F_1(x,y) &= \sqrt{4 - (x-0.5)^2 - (y-0.5)^2}, \\
F_2(x,y) &= \tanh(9y - 9x), \\
F_3(x,y) &= \frac{1.25 + \cos(5.4y)}{6 + 6(3x-1)^2}, \\
F_4(x,y) &= \exp\left(-\frac{81}{16}\left[(x-0.5)^2 + (y-0.5)^2\right]\right).
\end{aligned}$$

Let $e_i(F_j, n)$ be the maximal error of the approximated mean curvature computed by $\Delta_M^{(i)}$ over the above-mentioned local triangulations with $r = 1/n$ and the exact mean curvature computed from the continuous surfaces defined by F_j. Table 1 shows the maximal error $e_2(F_j, n)$. The results show that $\Delta_M^{(2)}$ is an approximation of LB operator. However, the degree of the approximation is not improved as $n \to \infty$. Hence, *$\Delta_M^{(2)} p$ is not a discrete LB operator that converges to the exact value.* Notice that the approximation error is rather large for function F_2.

Table 2 shows the maximal error $e_3(F_j, n)$. The results show that the approximation errors approach to zero at the rate about 1/4 as $r \to 0$ at the rate 1/2. Hence, *discrete LB operator $\Delta_M^{(3)} p$ converges quadratically to the exact mean curvature.*

Table 3 shows the maximal error $e_4(F_j, n)$. As the results shown in Table 2, *the expected convergent property is observed for $\Delta_M^{(4)} p$.* Comparing the results in these two tables, we note that $\Delta_M^{(4)}$ is more accurate than $\Delta_M^{(3)}$ in most of the cases. But this is not always true. To

Table 1. The maximal errors $e_2(F_j, n)$.

n	8	16	32	64	128	256	512
$e_2(F_1, n)$	8.852e − 02	8.859e − 02	8.860e − 02	8.861e − 02	8.861e − 02	8.861e − 02	8.861e − 02
$e_2(F_2, n)$	5.189e − 00	7.241e − 00	8.191e − 00	8.495e − 00	8.573e − 00	8.593e − 00	8.598e − 00
$e_2(F_3, n)$	4.490e − 01	6.210e − 01	7.658e − 01	8.216e − 01	8.358e − 01	8.393e − 01	8.402e − 01
$e_2(F_4, n)$	1.306e − 00	9.679e − 01	1.277e − 00	1.454e − 00	1.499e − 00	1.510e − 00	1.513e − 00

Table 2. The maximal errors $e_3(F_j, n)$.

n	8	16	32	64	128	256	512
$e_3(F_1, n)$	2.072e − 04	5.174e − 05	1.293e − 05	3.232e − 06	8.075e − 07	2.014e − 07	4.994e − 08
$e_3(F_2, n)$	2.951e − 00	1.328e − 00	3.938e − 01	1.029e − 01	2.602e − 02	6.523e − 03	1.632e − 03
$e_3(F_3, n)$	8.965e − 01	2.678e − 01	7.043e − 02	1.784e − 02	4.475e − 03	1.120e − 03	2.800e − 04
$e_3(F_4, n)$	2.623e − 00	8.424e − 01	2.261e − 01	5.757e − 02	1.446e − 02	3.619e − 03	9.050e − 04

illustrate this, we take a different triangulation of the domain as shown in Figure 1b, where we take $q_1 = r(1, 0)$, $q_2 = r(\sqrt{2}/2, \sqrt{2}/2)$, $q_3 = r(\sqrt{2}/2 - 1, \sqrt{2}/2)$, and $q_{k+3} = -q_k$, $k = 1, 2, 3$. Both $\Delta_M^{(3)}$ and $\Delta_M^{(4)}$ are convergent quadratically for this domain triangulation. Table 4 shows the ratios of the maximal errors of $\Delta_M^{(4)}$ and $\Delta_M^{(3)}$. It is observed that $\Delta_M^{(4)}$ is worse than $\Delta_M^{(3)}$ in most cases. This may be unexpected to the authors of [10], considering $\Delta_M^{(4)}$ is developed later and more elaborate than $\Delta_M^{(3)}$.

The convergence property of $\Delta_M^{(3)}$ and $\Delta_M^{(4)}$ holds only for very special triangulation of surfaces (see Figure 2). To illustrate this, we now perturb θ_4, θ_5, θ_6 in (3.1) by 1%. That is, we time θ_4, θ_5 and θ_6 by factors $1 + 0.01$, $1 - 0.01$ and $1 + 0.01$, respectively. Table 5 shows the corresponding results to Table 2. Here, no convergence property is observed. This observation is true also for $\Delta_M^{(4)}$.

Table 3. The maximal errors $e_4(F_j, n)$.

n	8	16	32	64	128	256	512
$e_4(F_1, n)$	5.751e − 05	1.435e − 05	3.585e − 06	8.961e − 07	2.240e − 07	5.597e − 08	1.405e − 08
$e_4(F_2, n)$	3.043e − 00	1.319e − 00	3.013e − 01	7.449e − 02	1.895e − 02	4.757e − 03	1.190e − 03
$e_4(F_3, n)$	7.204e − 01	2.091e − 01	5.454e − 02	1.378e − 02	3.456e − 03	8.645e − 04	2.162e − 04
$e_4(F_4, n)$	1.865e − 00	5.483e − 01	1.430e − 01	3.612e − 02	9.054e − 03	2.265e − 03	5.663e − 04

Table 4. The ratios of maximal errors $e_4(F_j, n)$ and $e_3(F_j, n)$.

n	8	16	32	64	28	256	512
$e_4(F_1, n)/e_3(F_1, n)$	1.663	1.668	1.669	1.674	1.669	1.670	1.670
$e_4(F_2, n)/e_3(F_2, n)$	0.933	1.066	1.061	1.080	1.086	1.087	1.088
$e_4(F_3, n)/e_3(F_3, n)$	1.012	1.160	1.354	1.362	1.363	1.364	1.364
$e_4(F_4, n)/e_3(F_4, n)$	0.917	1.206	1.411	1.429	1.433	1.434	1.435

Table 5. The maximal errors $e_3(F_j, n)$ for perturbed data.

n	8	16	32	64	128	256	512
$e_3(F_1, n)$	1.600e − 03	1.734e − 03	1.766e − 03	1.774e − 03	1.775e − 03	1.775e − 03	1.775e − 03
$e_3(F_2, n)$	3.030e − 00	1.446e − 00	5.004e − 01	1.995e − 01	1.649e − 01	1.558e − 01	1.530e − 01
$e_3(F_3, n)$	8.751e − 01	2.431e − 01	5.107e − 02	3.920e − 02	3.621e − 02	3.546e − 02	3.527e − 02
$e_3(F_4, n)$	2.614e − 00	8.274e − 01	2.083e − 01	4.184e − 02	4.228e − 02	4.243e − 02	4.249e − 02

All these results shown in Tables 1–5 are obtained using the regular mesh. That is, each of the vertices has valence 6. We also test the case where the valence of the vertex is other than 6. No convergent property is observed for those discrete LB operators.

In the next section, we will give conditions under which $\Delta_M^{(3)}$ and $\Delta_M^{(4)}$ converge to the exact value.

4. CONVERGENCE OF LB OPERATORS

In the previous section, we have shown that the discrete LB operators defined by (2.8) and (2.9) converge numerically in some special cases. In this section, we give sufficient conditions for the convergence. The proof of the convergence results are given in the next section.

THEOREM 4.1. *Let p be a vertex of M with valence 6, and let $p_1, \dots, p_6$ be its neighbor vertices. Suppose p and p_i $(i = 1, \dots, 6)$ are on a sufficiently smooth parametric surface $F(\xi_1, \xi_2) \in \mathbb{R}^3$, and there exist $q, q_1, \dots, q_6 \in \mathbb{R}^2$, such that*

$$p = F(q) \quad p_i = F(q_i) \quad \text{and} \quad q_i = q_{i-1} + q_{i+1} - q, \qquad i = 1, \dots, 6. \tag{4.1}$$

Then,

$$K(p, r) = H(p) + O(r^2), \qquad \text{as } r \to 0,$$

where $H(p)$ is the mean curvature vector at p,

$$K(p, r) = \frac{3}{2A(p, r)} \sum_{i=1}^{6} \frac{\cot \alpha_i(r) + \cot \beta_i(r)}{2} [p_i(r) - p], \tag{4.2}$$

$$p_i(r) = F(q_i(r)), \quad q_i(r) = q + r(q_i - q), \qquad i = 1, \dots, 6, \tag{4.3}$$

and $A(p, r)$ is the sum of the areas of triangle $[p, p_i(r), p_{i+1}(r)]$, $\alpha_i(r)$ and $\beta_i(r)$ are defined as in (2.9) from vertices $p_j(r)$.

The proof of the theorem is rather meticulous and lengthy, so we put it into a separate section (Section 5). The basic idea of the proof is to expand $K(p, r)$ into a power series with respect to r. Theorem 4.1 says that $\Delta_M^{(3)}$ converges in the rate $O(r^2)$. For discrete LB operator $\Delta_M^{(4)}$, a similar result could be obtained.

THEOREM 4.2. *Under the conditions of Theorem 4.1, we have*

$$\frac{1}{2A_M(p, r)} \sum_{i=1}^{6} \frac{\cot \alpha_i(r) + \cot \beta_i(r)}{2} [p_i(r) - p] = H(p) + O(r^2), \qquad \text{as } r \to 0,$$

where $A_M(p, r)$, $\cot \alpha_i(r)$ and $\cot \beta_i(r)$ are defined as in (2.9) from vertices $p_j(r)$.

Though we have the relation $\Delta_{\mathcal{M}} p = 2H(p)$, the convergence of $\Delta_M^{(i)} p$ $(i = 3, 4)$ do not equal the convergence of $\Delta_M^{(i)} f(p)$ $(i = 3, 4)$, where f is a smooth function on $\mathcal{M}$. However, similar convergence results can be proved indeed for $\Delta_M^{(i)} f$.

THEOREM 4.3. *Let f be a sufficiently smooth function over surface $\mathcal{M}$. Then, under the conditions of Theorem 4.1, we have*

$$\lim_{r \to 0} \frac{3}{A(p, r)} \sum_{i=1}^{6} \frac{\cot \alpha_i(r) + \cot \beta_i(r)}{2} [f(p_i(r)) - f(p)] = \Delta_{\mathcal{M}} f(p), \tag{4.4}$$

$$\lim_{r \to 0} \frac{1}{A_M(p, r)} \sum_{i=1}^{6} \frac{\cot \alpha_i(r) + \cot \beta_i(r)}{2} [f(p_i(r)) - f(p)] = \Delta_{\mathcal{M}} f(p). \tag{4.5}$$

Using more detail derivation, we can show that the convergence rate of (4.4) and (4.5) is also quadratic.

REMARK 1. Notice that the convergence results are obtained under particular conditions. This particularity is not only in the position the vertices locate, but also in the valence the vertices have. However, this particular case is very useful and important, because many numerical simulations of geometric partial differential equations are conducted over a triangulated domain formed by a uniform three-directional partition. This kind of domain triangulation satisfies the conditions of Theorem 4.1.

REMARK 2. From the proof of Theorem 4.1, we can see that there are many term cancellations. These cancellations happen because the relations $q_{i+3} + q_i = 2q$ $(i = 1, 2, 3)$ holds. In the general case, such cancellations could not happen. Hence, to get a convergence result for the general case, two ring data around each vertex may be necessarily involved instead of one.

5. PROOFS OF THE CONVERGENCE RESULTS

PROOF OF THEOREM 4.1. Without loss of generality, we may assume that q is the origin of the $\xi_1\xi_2$-plane, and $q_1 = (1, 0)$ (see Figure 3a). Then, there exists a constant $a > 0$ (which is the length of $q_2 - q$) and an angle θ, such that

$$q_2 = a\,(\cos\theta, \sin\theta)\,,$$

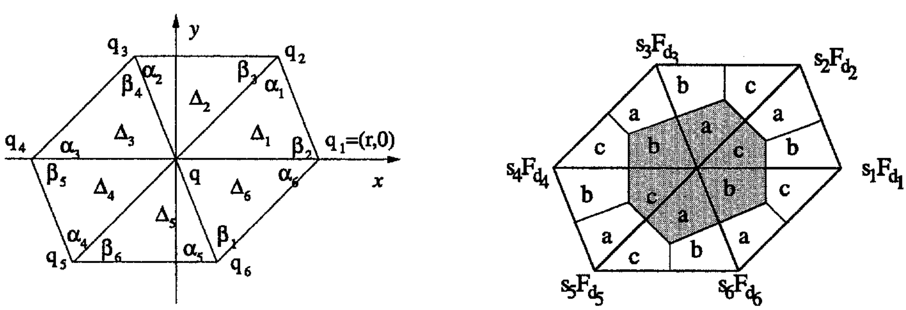
then, $q_3 = q_2 - q_1 = (a\cos\theta - 1, a\sin\theta)$ and $q_{i+3} = -q_i$, $i = 1, 2, 3$. Let

$$q_i = s_i d_i, \qquad \text{with } s_i = \|q_i\|,\ d_i = q_i/\|q_i\|, \quad i = 1, 2, \dots, 6.$$

Then, $s_1 = 1$, $s_2 = a$, $s_3 = \sqrt{a^2 - 2a\cos\theta + 1}$, $s_{i+3} = s_i$, $i = 1, 2, 3$. Now, we expand $K(p, r)$ into the form

$$K\,(p, r) = A_0 + A_1 r + O\left(r^2\right),$$

and show that A_0 is the mean curvature $H(p)$ and $A_1 = 0$.



(a) The triangulation of the domain D.

(b) Space triangles $[0, s_iF_{d_i}, s_{i+1}F_{d_{i+1}}]$, for $i = 1, \dots, 6$, and areas $\lim_{r\to 0} A(p, r)/r^2$ (the total) and $\lim_{r\to 0} A_M(p, r)/r^2$ (the dark part).

Figure 3.

Let $F_{d_i}^j := F_{d_i}^j(q)$ denote the directional derivative of F at q of order j and in the direction d_i. Then, we have, by Taylor expansion,

$$p_i\,(r) = F\,(q_i\,(r)) = p + s_i r F_{d_i} + \frac{1}{2} s_i^2 r^2 F_{d_i}^2 + \frac{1}{6} s_i^3 r^3 F_{d_i}^3 + O\left(r^4\right), \qquad i = 1, \dots, 6. \tag{5.1}$$

Let $\Delta_i(r)$ denote the area of the triangle $[p, p_i(r), p_{i+1}(r)]$, where the indices are to be understood modulo 6. Then, using the area formula: $\Delta(u,v) = 1/2\sqrt{\|u\|^2\|v\|^2 - \langle u,v\rangle^2}$ for the triangle formed by two vectors u and v in $\mathbb{R}^3$, we have

$$\begin{aligned}
2\Delta_i(r) &= \sqrt{\|p_i(r)-p\|^2\|p_{i+1}(r)-p\|^2 - \langle p_i(r)-p,\ p_{i+1}(r)-p\rangle^2} \\
&= r^2\sqrt{\delta_i^{(0)} + \delta_i^{(1)} r + \delta_i^{(2)} r^2 + O(r^3)} \\
&= \Delta_i^{(0)} r^2 + \Delta_i^{(1)} {\delta_i^{(0)}}^{-1/2} r^3 + \Delta_i^{(2)} {\delta_i^{(0)}}^{-3/2} r^4 + O(r^5),
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
\delta_i^{(0)} &= s_i^2 s_{i+1}^2 \left[\|F_{d_i}\|^2 \left\|F_{d_{i+1}}\right\|^2 - \left\langle F_{d_i}, F_{d_{i+1}}\right\rangle^2\right], \\
\delta_i^{(1)} &= s_i^3 s_{i+1}^2 \left[\left\|F_{d_{i+1}}\right\|^2 \left\langle F_{d_i}, F_{d_i}^2\right\rangle - \left\langle F_{d_i}, F_{d_{i+1}}\right\rangle \left\langle F_{d_{i+1}}, F_{d_i}^2\right\rangle\right] \\
&\quad + s_i^2 s_{i+1}^3 \left[\|F_{d_i}\|^2 \left\langle F_{d_{i+1}}, F_{d_{i+1}}^2\right\rangle - \left\langle F_{d_i}, F_{d_{i+1}}\right\rangle \left\langle F_{d_i}, F_{d_{i+1}}^2\right\rangle\right], \\
\delta_i^{(2)} &= s_i^2 s_{i+1}^4 \left[\frac{1}{3}\|F_{d_i}\|^2 \left\langle F_{d_{i+1}}, F_{d_{i+1}}^3\right\rangle + \frac{1}{4}\|F_{d_i}\|^2 \left\|F_{d_{i+1}}^2\right\|^2\right. \\
&\quad \left. - \frac{1}{3}\left\langle F_{d_i}, F_{d_{i+1}}\right\rangle \left\langle F_{d_i}, F_{d_{i+1}}^3\right\rangle - \frac{1}{4}\left\langle F_{d_i}, F_{d_{i+1}}^2\right\rangle^2\right] \\
&\quad + s_i^3 s_{i+1}^3 \left[\left\langle F_{d_i}, F_{d_i}^2\right\rangle \left\langle F_{d_{i+1}}, F_{d_{i+1}}^2\right\rangle - \frac{1}{2}\left\langle F_{d_i}, F_{d_{i+1}}\right\rangle \left\langle F_{d_i}^2, F_{d_{i+1}}^2\right\rangle\right. \\
&\quad \left. - \frac{1}{2}\left\langle F_{d_i}, F_{d_{i+1}}^2\right\rangle \left\langle F_{d_i}^2, F_{d_{i+1}}\right\rangle\right] \\
&\quad + s_i^4 s_{i+1}^2 \left[\frac{1}{3}\left\|F_{d_{i+1}}\right\|^2 \left\langle F_{d_i}, F_{d_i}^3\right\rangle + \frac{1}{4}\left\|F_{d_i}^2\right\|^2 \left\|F_{d_{i+1}}\right\|^2\right. \\
&\quad \left. - \frac{1}{3}\left\langle F_{d_i}, F_{d_{i+1}}\right\rangle \left\langle F_{d_{i+1}}, F_{d_i}^3\right\rangle - \frac{1}{4}\left\langle F_{d_i}^2, F_{d_{i+1}}\right\rangle^2\right],
\end{aligned}$$

and, using the expansion

$$\sqrt{c_0 + c_1 r + c_2 r^2 + \cdots} = \sqrt{c_0} + \frac{c_1}{2\sqrt{c_0}} r + \frac{4c_0c_2 - c_1^2}{8c_0\sqrt{c_0}} r^2 + \cdots,$$

$$\Delta_i^{(0)} = \sqrt{\delta_i^{(0)}}, \qquad \Delta_i^{(1)} = \frac{\delta_i^{(1)}}{2}, \qquad \Delta_i^{(2)} = \frac{4\delta_i^{(0)}\delta_i^{(2)} - \delta_i^{(1)}\delta_i^{(1)}}{8}.$$

Since $s_{i+2}d_{i+2} = s_{i+1}d_{i+1} - s_i d_i$ (see (4.1)), we can derive that

$$\delta_{i+1}^{(0)} = \delta_i^{(0)}, \qquad i = 1, \ldots, 6.$$

Let

$$t_i = \frac{\partial F(q)}{\partial \xi_i}, \qquad t_{ij} = \frac{\partial^2 F(q)}{\partial \xi_i \partial \xi_j}, \qquad g_{ij} = \langle t_i, t_j\rangle, \qquad g_{ijk} = \langle t_i, t_{jk}\rangle.$$

It follows from

$$F_{d_1} = t_1, \qquad F_{d_2} = t_1\cos\theta + t_2\sin\theta,$$

that

$$\begin{aligned}
\delta_1^{(0)} &= s_1^2 s_2^2 \left[\|t_1\|^2 \|t_1\cos\theta + t_2\sin\theta\|^2 - \langle t_1, t_1\cos\theta + t_2\sin\theta\rangle^2\right] \\
&= a^2 \sin^2\theta \det(G) \\
&= \delta,
\end{aligned}$$

where $\det(G) = g_{11}g_{22} - g_{12}^2$. Using the fact that

$$s_{i+3} = s_i, \quad d_{i+3} = -d_i, \qquad i = 1,2,3, \tag{5.3}$$

we have

$$F_{d_{i+3}}^j = (-1)^j F_{d_i}^j, \qquad i = 1,2,3, \quad j = 1,2,3, \tag{5.4}$$

and therefore,

$$\Delta_{i+3}^{(j)} = (-1)^j \Delta_i^{(j)}, \qquad i = 1,2,3, \quad j = 0,1,2.$$

Hence,

$$A(p,r) = \sum_{i=1}^{6} \Delta_i(r) = 3r^2\sqrt{\delta} + 3r^4 E\sqrt{\delta} + O(r^5),$$

where $E = \sum_{i=1}^{6} \Delta_i^{(2)}/(6\delta^2)$.

Now, we compute $\cot\alpha_i(r) + \cot\beta_i(r)$. For two vectors $u, v \in \mathbb{R}^3$, let α be the angle between them. Then, $\cot\alpha$ is given by

$$\cot\alpha = \frac{\langle u,v\rangle}{\sqrt{\|u\|^2\|v\|^2 - (u,v)^2}} = \frac{\langle u,v\rangle}{2\Delta(u,v)}.$$

Hence, using (5.1) and (5.2),

$$\begin{aligned}
\cot\alpha_i(r) &= \frac{\langle p_{i+1}(r) - p,\ p_{i+1}(r) - p_i(r)\rangle}{2\Delta_i(r)} \\
&= \frac{r^2\alpha_i^{(0)} + r^3\alpha_i^{(1)} + r^4\alpha_i^{(2)} + O(r^5)}{2\Delta_i(r)} \\
&= \frac{B_i^{(0)}}{\sqrt{\delta}} + r\frac{B_i^{(1)}}{\delta\sqrt{\delta}} + r^2\frac{B_i^{(2)}}{\delta^2\sqrt{\delta}} + O(r^3),
\end{aligned}$$

where

$$\begin{aligned}
\alpha_i^{(0)} &= s_{i+1}^2\left\|F_{d_{i+1}}\right\|^2 - s_i s_{i+1}\left\langle F_{d_i}, F_{d_{i+1}}\right\rangle, \\
2\alpha_i^{(1)} &= 2s_{i+1}^3\left\langle F_{d_{i+1}}, F_{d_{i+1}}^2\right\rangle - s_i^2 s_{i+1}\left\langle F_{d_{i+1}}, F_{d_i}^2\right\rangle - s_i s_{i+1}^2\left\langle F_{d_i}, F_{d_{i+1}}^2\right\rangle, \\
6\alpha_i^{(2)} &= s_{i+1}^4\left[2\left\langle F_{d_{i+1}}, F_{d_{i+1}}^3\right\rangle + \frac{3}{2}\left\|F_{d_{i+1}}^2\right\|^2\right] - s_i s_{i+1}^3\left\langle F_{d_i}, F_{d_{i+1}}^3\right\rangle \\
&\quad - \frac{3}{2}s_i^2 s_{i+1}^2\left\langle F_{d_i}^2, F_{d_{i+1}}^2\right\rangle - s_i^3 s_{i+1}\left\langle F_{d_{i+1}}, F_{d_i}^3\right\rangle,
\end{aligned}$$

and, using the expansion

$$\frac{a_0 + a_1 r + a_2 r^2 + \cdots}{b_0 + b_1 r + b_2 r^2 + \cdots} = \frac{a_0}{b_0} + \frac{a_1 b_0 - a_0 b_1}{b_0^2} r + \frac{a_2 b_0^2 - a_1 b_0 b_1 + a_0 b_1^2 - a_0 b_0 b_2}{b_0^3} r^2 \ldots,$$

$$\begin{aligned}
2B_i^{(0)} &= \alpha_i^{(0)}, \qquad 2B_i^{(1)} = \alpha_i^{(1)}\delta - \alpha_i^{(0)}\Delta_i^{(1)}, \\
2B_i^{(2)} &= \alpha_i^{(2)}\delta^2 - \alpha_i^{(1)}\Delta_i^{(1)}\delta + \alpha_i^{(0)}\left[\left(\Delta_i^{(1)}\right)^2 - \Delta_i^{(2)}\right].
\end{aligned}$$

It follows from (5.3),(5.4) that

$$\alpha_{i+3}^{(j)} = (-1)^j \alpha_i^{(j)}, \qquad i = 1,2,3, \quad j = 0,1,2,$$

and therefore,

$$B_{i+3}^{(j)} = (-1)^j \, B_i^{(j)}, \qquad i = 1,2,3, \quad j = 0,1,2.$$

Similarly,

$$\begin{aligned}
\cot \beta_i(r) &= \frac{\langle p_{i-1}(r) - p, p_{i-1}(r) - p_i(r) \rangle}{2\Delta_{i-1}(r)} \\
&= \frac{r^2 \beta_i^{(0)} + r^3 \beta_i^{(1)} r^4 \beta_i^{(2)} + O(r^5)}{2\Delta_{i-1}(r)} \\
&= \frac{\tilde{B}_i^{(0)}}{\sqrt{\delta}} + r \frac{\tilde{B}_i^{(1)}}{\delta\sqrt{\delta}} + r^2 \frac{\tilde{B}_i^{(2)}}{\delta^2\sqrt{\delta}} + O(r^3),
\end{aligned}$$

where

$$\begin{aligned}
\beta_i^{(0)} &= s_{i-1}^2 \left\| F_{d_{i-1}} \right\|^2 - s_i s_{i-1} \left\langle F_{d_{i-1}}, F_{d_i} \right\rangle, \\
2\beta_i^{(1)} &= 2 s_{i-1}^3 \left\langle F_{d_{i-1}}, F_{d_{i-1}}^2 \right\rangle - s_i^2 s_{i-1} \left\langle F_{d_{i-1}}, F_{d_i}^2 \right\rangle - s_{i-1}^2 s_i \left\langle F_{d_i}, F_{d_{i-1}}^2 \right\rangle, \\
6\beta_i^{(2)} &= s_{i-1}^4 \left[2 \left\langle F_{d_{i-1}}, F_{d_{i-1}}^3 \right\rangle + \frac{3}{2} \left\| F_{d_{i-1}}^2 \right\|^2 \right] - s_i s_{i-1}^3 \left\langle F_{d_i}, F_{d_{i-1}}^3 \right\rangle \\
&\quad - \frac{3}{2} s_i^2 s_{i-1}^2 \left\langle F_{d_{i-1}}^2, F_{d_i}^2 \right\rangle - s_i^3 s_{i-1} \left\langle F_{d_{i-1}}, F_{d_i}^3 \right\rangle,
\end{aligned}$$

and

$$\begin{aligned}
&2\tilde{B}_i^{(0)} = \beta_i^{(0)}, \qquad 2\tilde{B}_i^{(1)} = \beta_i^{(1)} \delta - \beta_i^{(0)} \Delta_{i-1}^{(1)}, \\
&2\tilde{B}_i^{(2)} = \beta_i^{(2)} \delta^2 - \beta_i^{(1)} \Delta_{i-1}^{(1)} \delta + \beta_{i-1}^{(0)} \left[\left(\Delta_{i-1}^{(1)} \right)^2 - \Delta_{i-1}^{(2)} \right].
\end{aligned}$$

It follows from (5.3),(5.4) that

$$\beta_{i+3}^{(j)} = (-1)^j \, \beta_i^{(j)}, \qquad i = 1,2,3, \quad j = 0,1,2,$$

and therefore,

$$\tilde{B}_{i+3}^{(j)} = (-1)^j \, \tilde{B}_i^{(j)}, \qquad i = 1,2,3, \quad j = 0,1,2.$$

Hence, the coefficients in (4.2) are given by

$$\begin{aligned}
w_i &:= \frac{3}{4} \frac{\cot \alpha_i(r) + \cot \beta_i(r)}{A(p,r)} \qquad (5.5) \\
&= \frac{1}{4} \frac{\left(B_i^{(0)} + \tilde{B}_i^{(0)} \right) \delta^{-1/2} + r \left(B_i^{(1)} + \tilde{B}_i^{(1)} \right) \delta^{-3/2} + r^2 \left(B_i^{(2)} + \tilde{B}_i^{(2)} \right) \delta^{-5/2} + O\left(r^3\right)}{r^2 \delta^{1/2} + r^4 E \delta^{1/2} + O\left(r^5\right)} \\
&= \frac{1}{4r^2 \det(G)^2} \left[w_i^{(0)} + r w_i^{(1)} + r^2 w_i^{(2)} + O\left(r^3\right) \right]
\end{aligned}$$

with (note that $\delta = a^2 \sin^2\theta \det(G)$)

$$\begin{aligned}
w_i^{(0)} &= \frac{\left(B_i^{(0)} + \tilde{B}_i^{(0)} \right) \delta}{a^4 \sin^4\theta}, \\
w_i^{(1)} &= \frac{\left(B_i^{(1)} + \tilde{B}_i^{(1)} \right)}{a^4 \sin^4\theta}, \\
w_i^{(2)} &= \frac{\left(B_i^{(2)} + \tilde{B}_i^{(2)} \right) \delta^{-1} - \left(B_i^{(0)} + \tilde{B}_i^{(0)} \right) E\delta}{a^4 \sin^4\theta},
\end{aligned}$$

and

$$w_{i+3}^{(j)} = (-1)^j \, w_i^{(j)}, \qquad i = 1,2,3, \quad j = 0,1,2. \qquad (5.6)$$

Therefore,

$$\begin{aligned}K(p,r) &= \frac{1}{4\det(G)^2 r^2}\sum_{i=1}^{3}\{w_i[p_i(r)-p]+w_{i+3}[p_{i+3}(r)-p]\}\\
&= \frac{1}{4\det(G)^2}\sum_{i=1}^{3}\left\{w_i^{(0)}\frac{p_i(r)+p_{i+3}(r)-2p}{r^2}+w_i^{(1)}\frac{p_i(r)-p_{i+3}(r)}{r}\right.\\
&\quad\left.+w_i^{(2)}[p_i(r)+p_{i+3}(r)-2p]\right\}+O(r^2)\\
&= \frac{1}{4\det(G)^2}\sum_{i=1}^{3}\left[w_i^{(0)}s_i^2F_{d_i}^2+2w_i^{(1)}s_iF_{d_i}\right]+O(r^2).\end{aligned}$$

Denote

$d_i=(\gamma_i,\lambda_i)$, i.e., $(\gamma_1,\lambda_1)=(1,0),(\gamma_2,\lambda_2)=(\cos\theta,\sin\theta),(\gamma_3,\lambda_3)=(a\cos\theta-1,a\sin\theta)/s_3$.

Then, we have

$$\begin{aligned}F_{d_i}&=\gamma_it_1+\lambda_it_2,\\ F_{d_i}^2&=\gamma_i^2t_{11}+2\gamma_i\lambda_it_{12}+\lambda_i^2t_{22},\end{aligned}\tag{5.7}$$

hence,

$$\begin{aligned}4\det(G)^2K(p,r)&=2\left[w_1^{(1)}s_1+w_2^{(1)}s_2\gamma_2+w_3^{(1)}s_3\gamma_3\right]t_1\\
&\quad+2\left[w_2^{(1)}s_2\lambda_2+w_3^{(1)}s_3\lambda_3\right]t_2+\left[w_1^{(0)}s_1^2+w_2^{(0)}s_2^2\gamma_2^2+w_3^{(0)}s_3^2\gamma_3^2\right]t_{11}\\
&\quad+2\left[w_2^{(0)}s_2^2\gamma_2\lambda_2+w_3^{(0)}s_3^2\gamma_3\lambda_3\right]t_{12}+\left[w_2^{(0)}s_2^2\lambda_2^2+w_3^{(0)}s_3^2\lambda_3^2\right]t_{22}+O(r^2).\end{aligned}\tag{5.8}$$

Denote the right-handed side of (5.8) as

$$c_1t_1+c_2t_2+c_{11}t_{11}+c_{12}t_{12}+c_{22}t_{22}+O(r^2).\tag{5.9}$$

Then, using Mathematica to conduct this derivation, we could derive that

$$\begin{aligned}c_1&=2\left[-2g_{12}^2g_{212}-g_{22}(g_{11}g_{122}+g_{22}g_{111})+g_{12}(2g_{22}g_{112}+g_{22}g_{211}+g_{11}g_{222})\right],\\
c_2&=-2\left[2g_{12}^2g_{112}-g_{11}g_{12}g_{111}+g_{11}(-g_{12}g_{122}-2g_{12}g_{212}+g_{22}g_{211}+g_{11}^2g_{222})\right],\\
c_{11}&=2g_{22}(g_{11}g_{22}-g_{12}^2),\\
c_{12}&=-4g_{12}(g_{11}g_{22}-g_{12}^2),\\
c_{22}&=2g_{11}(g_{11}g_{22}-g_{12}^2).\end{aligned}\tag{5.10}$$

Hence,

$$\begin{aligned}&2\det(G)^2K(p,r)\\
&=g_{22}\det(G)t_{11}+g_{11}\det(G)t_{22}-2g_{12}\det(G)t_{12}+O(r^2)\\
&\quad-[g_{22}(g_{22}g_{111}-g_{12}g_{211})+g_{11}(g_{22}g_{122}-g_{12}g_{222})-2g_{12}(g_{22}g_{112}-g_{12}g_{212})]t_1\\
&\quad-[g_{22}(g_{11}g_{211}-g_{12}g_{111})+g_{11}(g_{11}g_{222}-g_{12}g_{122})-2g_{12}(g_{11}g_{212}-g_{12}g_{112})]t_2\\
&=\det(G)[g_{22}t_{11}+g_{11}t_{22}-2g_{12}t_{12}]-[t_1,t_2]\left\{g_{22}\begin{bmatrix}g_{22}&-g_{12}\\-g_{12}&g_{11}\end{bmatrix}\begin{bmatrix}g_{111}\\g_{211}\end{bmatrix}\right.\\
&\quad\left.+g_{11}\begin{bmatrix}g_{22}&-g_{12}\\-g_{12}&g_{11}\end{bmatrix}\begin{bmatrix}g_{122}\\g_{222}\end{bmatrix}-2g_{12}\begin{bmatrix}g_{22}&-g_{12}\\-g_{12}&g_{11}\end{bmatrix}\begin{bmatrix}g_{112}\\g_{212}\end{bmatrix}\right\}+O(r^2)\\
&=\det(G)[g_{22}t_{11}+g_{11}t_{22}-2g_{12}t_{12}]\\
&\quad-\det(G)[t_1,t_2]G^{-1}\left\{g_{22}\begin{bmatrix}g_{111}\\g_{211}\end{bmatrix}+g_{11}\begin{bmatrix}g_{122}\\g_{222}\end{bmatrix}-2g_{12}\begin{bmatrix}g_{112}\\g_{212}\end{bmatrix}\right\}+O(r^2)\\
&=\det(G)[g_{22}t_{11}+g_{11}t_{22}-2g_{12}t_{12}]\\
&\quad-\det(G)[t_1,t_2]G^{-1}[t_1,t_2]^\top[g_{22}t_{11}+g_{11}t_{22}-2g_{12}t_{12}]+O(r^2)\\
&=\det(G)\left(I-[t_1,t_2]G^{-1}[t_1,t_2]^\top\right)[g_{22}t_{11}+g_{11}t_{22}-2g_{12}t_{12}]+O(r^2).\end{aligned}$$

Therefore,

$$K(p,r) = \frac{\left(I - [t_1,t_2]\, G^{-1} [t_1,t_2]^\top\right) [g_{22}t_{11} + g_{11}t_{22} - 2g_{12}t_{12}]}{2\det(G)} + O\left(r^2\right).$$

The first term of the right-handed side of (5.11) is the mean curvature vector (see [23]). Hence, Theorem 4.1 is proved. ∎

PROOF OF THEOREM 4.2. In fact, the difference of the two discrete LB operators is the areas $A(p_i)$ and $A_M(p_i)$. Hence, we only need to show

$$\frac{A(p,r)}{A_M(p,r)} = 3 + O\left(r^2\right), \qquad \text{as } r \to 0, \tag{5.12}$$

where $A_M(p,r)$ is defined as $A_M(p_i)$ in (2.9) from the vertices $p_j(r)$ defined by (4.3). (5.12) could be proved using the expansion (5.1). Without loss of generality, we may assume p is the origin. Then, we have

$$[p, p_i(r), p_{i+1}(r)] = r\left[0, s_i F_{d_i}, s_{i+1}F_{d_{i+1}}\right] + O\left(r^2\right).$$

Using conditions (4.1) and (5.3), we can see that the space triangles $[0, s_iF_{d_i}, s_{i+1}F_{d_{i+1}}]$ ($i = 1,\dots,6$) are congruent (see Figure 3b). In Figure 3b, each triangle is divided into three parts by its circumcenter, whose areas are denoted by a, b, and c, respectively. The total area is $6(a+b+c)$. The area of the dark part is $2(a+b+c)$. Hence, (5.12) is true. ∎

PROOF OF THEOREM 4.3. To prove (4.4), we first extend the function f smoothly to a neighborhood of surface $\mathcal{M}$, so that f could be regarded as a trivariate function over a 3D domain. Obviously, such an extension exists. Let ∇ be the classical gradient operator acting on trivariate functions. Then, using the relations

$$\left[\frac{\partial f(p)}{\partial \xi_1}, \frac{\partial f(p)}{\partial \xi_2}\right]^\top = [t_1,t_2]^\top \nabla f(p), \qquad \frac{\partial \nabla f(p)}{\partial \xi_1} = \nabla^2 f(p)\, t_1, \qquad \frac{\partial \nabla f(p)}{\partial \xi_2} = \nabla^2 f(p)\, t_2,$$

we can rewrite (2.1) into the following form

$$\Delta_{\mathcal{M}} f(p) = 2H(p)^\top \nabla f(p) + \frac{(g_{22}t_1 - g_{12}t_2)^\top \nabla^2 f(p)\, t_1 + (g_{11}t_2 - g_{12}t_1)^\top \nabla^2 f(p)\, t_2}{\det(G)}. \tag{5.13}$$

Now, we compute the left-handed side of (4.4) and show that it equals to the right-handed side of (5.13). Using the relation

$$f(p_j(r)) = f(p_i) + (p_j(r) - p_i)^\top \nabla f(p_i) + \frac{1}{2}(p_j(r) - p_i)^\top \nabla^2 f(p_i)(p_j(r) - p_i) + O\left(r^3\right),$$

and the relations (5.5), (5.6), and (5.7), we can write the left-handed side of (4.4) as

$$\begin{aligned}
&\lim_{r\to 0} 2\sum_{i=1}^{6} w_i(p_j(r) - p_i)^\top \nabla f(p_i) + \lim_{r\to 0} 2r^2 \sum_{i=1}^{3} w_i s_i^2 F_{d_i}^\top \nabla^2 f F_{d_i} \\
&\qquad = 2H(p)^\top \nabla f(p_i) + \lim_{r\to 0} \frac{[c_{11}, c_{12}/2][t_1,t_2]^\top \nabla^2 f(p_i) t_1 + [c_{12}/2, c_{22}][t_1,t_2]^\top \nabla^2 f(p_i) t_2}{2\det(G)^2},
\end{aligned}$$

where c_{ij} are defined in (5.9). Using (5.10), we obtain the right-handed side of (5.13). Hence, (4.4) is proven. (4.5) follows from (4.4) and (5.12). ∎

6. CONCLUSIONS

We have reviewed several existing discretizations of LB operators and shown that most of them are not convergent. Only two of them, which are proposed by Desbrun *et al.* and Meyer *et al.*, converge for some special cases. We also point out that these special case are very useful and therefore, very important. Currently, we are looking for discrete LB operators which converge under more general conditions.

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